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Controllability–observability of expanded composite systems

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Abstract

The relation between original and expanded systems within the Inclusion Principle from the point of view of controllability–observability of both subsystems and composite systems is studied. It is proved that complementary matrices always exist ensuring that the subsystems and the overall expanded system are simultaneously controllable–observable. Two practically important large classes of complementary matrices are identified to offer results computationally attractive. First, the existence of complementary matrices ensuring controllability–observability of decoupled subsystems is proved. Then, using this result, the same property is proved for the composite expanded system. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

The problem dealt with in this paper primarily appears in the study of controllability and observability of interconnected systems within the Inclusion Principle. The principle defines a framework for two dynamic systems with different dimensions in which solutions of the system with larger dimension include solutions of the system with smaller dimension. Both systems are related through linear transformations (expansions and contractions) that have the freedom of the selection of the so-called complementary matrices.

1.1. Relevant references

The Inclusion Principle has been developed by Šiljak and his co-workers [5,6,12]. The conditions given in previous works [5–8,12] on the complementary matrices have a fundamental, implicit nature, in the sense that it is not easy to select specific values for these matrices. In fact, only two particular forms of aggregations and restrictions have been commonly adopted in the literature for numerical computations [1,7,12,14]. A new characterization of the complementary matrices has been recently presented in [2,3,9], which gives a more explicit way for their selection and which includes aggregations and restrictions as particular cases. It relies on a new constructive way of approaching the concept of canonical form within the Inclusion Principle previously proposed in [6,12].

One of the research issues within the Inclusion Principle is the question of whether structural properties of the systems are transmitted or not when expansions and/or contractions are performed. In this sense, when using the particular forms of complementary matrices used in [8], an original system that is controllable and observable becomes either controllable or observable in its expanded form but not simultaneously controllable–observable, but this only concerns the overall systems. It means that no results are available when considering controllability–observability of subsystems between original and expanded composite systems. It is well known that the controllability of disjoint subsystems is not sufficient for controllability of the composite system [4,11,12]. Consequently, the study of controllability–observability on subsystems level must simultaneously include the study on the overall system level.

1.2. Outline of the paper

The result contributed by this paper is that an expanded system can always preserve controllability–observability of both the subsystems and the overall system provided that the original system holds both the properties when considering a disjoint structure of its subsystems.

The paper is organized as follows. Section 2 states the problem, first including necessary preliminaries on the Inclusion Principle in Section 2.1 and on the expansion–contraction process with the key structure of the complementary matrices in Section 2.2. Two practical important broad classes of complementary matrices, from the computational viewpoint, are specified in Section 2.3. The problem is formulated in Section 2.4. Section 3 presents the main result on the preservation of controllability–observability for one of these classes. Section 3.1 presents the results for individual subsystems when neglecting interconnections, while Section 3.2 employs the controllability–observability results for decoupled subsystems from the preceding part to prove the preservation of both these properties for the composite expanded system.

2. Problem formulation

2.1. Preliminaries: the Inclusion Principle

Consider a pair of linear systems

$$\mathbf{S}: \quad \begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \qquad \tilde{\mathbf{S}}: \quad \begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \\ \tilde{y} &= \tilde{C}\tilde{x}, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^l$ are the state, input, output of $\mathbf{S}$ at time $t \in \mathbb{R}^+$, and $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$, $\tilde{u}(t) \in \mathbb{R}^{\tilde{m}}$, $\tilde{y}(t) \in \mathbb{R}^{\tilde{l}}$ are those ones of $\tilde{\mathbf{S}}$. A, B, C and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are constant matrices of dimensions $n \times n$, $n \times m$, $l \times n$ and $\tilde{n} \times \tilde{n}$, $\tilde{n} \times \tilde{m}$, $\tilde{l} \times \tilde{n}$, respectively. Suppose that the dimensions of the state, input, output vectors x, u, y of $\mathbf{S}$ are smaller than (or at most equal to) those of $\tilde{x}$, $\tilde{u}$, $\tilde{y}$ of $\tilde{\mathbf{S}}$. Denote $x(t; x_0, u)$ and $y[x(t)]$ the state behavior and the corresponding output of $\mathbf{S}$ for a fixed input $u(t)$ and for an initial state $x(0) = x_0$, respectively. Similar notations $\tilde{x}(t; \tilde{x}_0, \tilde{u})$ and $\tilde{y}[\tilde{x}(t)]$ are used for the state behavior and output of system $\tilde{\mathbf{S}}$.

The systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are related by the following transformations $\tilde{x} = Vx$, $x = U\tilde{x}$, $\tilde{u} = Ru$, $u = Q\tilde{u}$, $\tilde{y} = Ty$, $y = S\tilde{y}$, where V, R and T are constant matrices of appropriate dimensions and full column ranks. U, Q and S are constant matrices of appropriate dimensions and full row ranks satisfying the relations $UV = I_n$, $QR = I_m$, $ST = I_l$ where I_n, I_m, I_l are identity matrices of indicated dimensions.

Definition 1 (*Inclusion Principle*). We say that the system $\tilde{\mathbf{S}}$ includes the system $\mathbf{S}$, that is, $\mathbf{S}$ is included by $\tilde{\mathbf{S}}$, if there exists a quadruplet (U, V, R, S) such that, for any initial state x_0 and any fixed input $u(t)$ of $\mathbf{S}$, the choice $\tilde{x}_0 = Vx_0$, $\tilde{u}(t) = Ru(t)$ for all $t \geqslant 0$ of the initial state $\tilde{x}_0$ and input $\tilde{u}(t)$ of the system $\tilde{\mathbf{S}}$, implies $x(t; x_0, u) = U\tilde{x}(t; \tilde{x}_0, \tilde{u})$, $y[x(t)] = S\tilde{y}[\tilde{x}(t)]$ for all $t \geqslant 0$.

Definition 2. If the system $\tilde{\mathbf{S}}$ includes the system $\mathbf{S}$, then $\tilde{\mathbf{S}}$ it is said to be an expansion of $\mathbf{S}$ and $\mathbf{S}$ is a contraction of $\tilde{\mathbf{S}}$.

Definition 1 implies that the system $\tilde{\mathbf{S}}$ contains all the necessary information about the behavior of the system $\mathbf{S}$. We can extract properties of $\mathbf{S}$ from $\tilde{\mathbf{S}}$. Suppose that the pairs of matrices (U, V), (Q, R) and (S, T) are given. Then, the matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ can be expressed as

$$\tilde{A} = VAU + M, \quad \tilde{B} = VBQ + N, \quad \tilde{C} = TCU + L, \tag{2}$$

where M, N and L are *complementary matrices* of appropriate dimensions. For $\tilde{\mathbf{S}}$ to be an expansion of $\mathbf{S}$, a proper choice of M, N and L is required, which is provided by the following theorem.

Theorem 3. *The system* $\tilde{\mathbf{S}}$ *is an expansion of the system* $\mathbf{S}$ *if and only if*

$$UM^iV = 0, \quad UM^{i-1}NR = 0, \quad SLM^{i-1}V = 0, \quad SLM^{i-1}NR = 0 \tag{3}$$

for all $i = 1, \ldots, \tilde{n}$.

The application of the expansion–contraction process summarized above requires the specific selection of the transformation matrices. It includes the choice of particular complementary matrices to satisfy Theorem 3.

2.2. Complementary matrices

In order to simplify the notation, consider $\mathbf{S}$ as a composite system composed of disjoint subsystems (A_{ii}, B_{ii}, C_{ii}), $i = 1, 2, 3$, with interconnections A_{ij}, B_{ij}, C_{ij}, $j = 1, 2, 3$, $i \neq j$, with the following structure:

$$\mathbf{S}: \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{array}{cc} & \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} & \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$+ \begin{array}{cc} & \begin{array}{ccc} m_1 & m_2 & m_3 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} & \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \end{array} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{4}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{array}{cc} & \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{array}{c} l_1 \\ l_2 \\ l_3 \end{array} & \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where n_i, m_i and l_i indicate the dimensions of the corresponding matrices with $n_1 + n_2 + n_3 = n$, $m_1 + m_2 + m_3 = m$, $l_1 + l_2 + l_3 = l$, $n_1 + 2n_2 + n_3 = \tilde{n}$, $m_1 +$

$2m_2 + m_3 = \tilde{m}$ and $l_1 + 2l_2 + l_3 = \tilde{l}$. This system can be simultaneously decomposed into two subsystems with an overlapped part denoted by dash lines.

Now consider the expansion of (4) via the following transformation matrices V, R, T:

$$V = \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{pmatrix}, \quad R = \begin{pmatrix} I_{m_1} & 0 & 0 \\ 0 & I_{m_2} & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{pmatrix},$$
$$T = \begin{pmatrix} I_{l_1} & 0 & 0 \\ 0 & I_{l_2} & 0 \\ 0 & I_{l_2} & 0 \\ 0 & 0 & I_{l_3} \end{pmatrix}. \tag{5}$$

From these matrices, we define $U = (V^{\mathrm{T}}V)^{-1}V^{\mathrm{T}}$, $Q = (R^{\mathrm{T}}R)^{-1}R^{\mathrm{T}}$, $S = (T^{\mathrm{T}}T)^{-1}T^{\mathrm{T}}$ as the pseudoinverses of V, R, T, respectively. Note that the particular form of the matrices V, R and T is not at all restrictive, but any other choice satisfying the conditions of the Inclusion Principle can be used. Structures (4) and (5) have been extensively adopted as a prototype overlapping decomposition structure in the literature and can be generalized for any number of interconnected overlapping subsystems [1–3,5–9,12,14].

Since the Inclusion Principle summarized before does not depend on the specific basis used in the state, input and output spaces for both systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$, we may introduce convenient changes of basis [6,12]. Having this idea in mind, we consider changes of basis in the spaces that correspond with the expanded system $\tilde{\mathbf{S}}$ [2,3,9,10]. The following theorem presents the form of the complementary matrices when we select initially the transformation matrices given in (5).

Theorem 4. *Consider the systems* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given in* (1). *Consider* $\mathbf{S}$ *with the form* (4) *and the transformation matrices* V, R *and* T *given in* (5). *Then*, $\tilde{\mathbf{S}}$ *includes* $\mathbf{S}$ *if and only if the following conditions are satisfied*:

$$\begin{cases} \begin{pmatrix} M_{12} \\ M_{23} + M_{33} \\ M_{42} \end{pmatrix} (M_{22} + M_{33})^{i-2} (M_{21} \quad M_{22} + M_{23} \quad M_{24}) = 0, \\ \begin{pmatrix} M_{12} \\ M_{23} + M_{33} \\ M_{42} \end{pmatrix} (M_{22} + M_{33})^{i-2} (N_{21} \quad N_{22} + N_{23} \quad N_{24}) = 0, \\ \begin{pmatrix} L_{12} \\ L_{23} + L_{33} \\ L_{42} \end{pmatrix} (M_{22} + M_{33})^{i-2} ((M_{21} \quad M_{22} + M_{23} \quad M_{24}) = 0, \end{cases}$$

for all $i = 2, \ldots, \tilde{n}$,

$$\left\{ \begin{pmatrix} L_{12} \\ L_{23} + L_{33} \\ L_{42} \end{pmatrix} (M_{22} + M_{33})^{i-2} (N_{21} \quad N_{22} + N_{23} \quad N_{24}) = 0 \right. \\ \textit{for all } \; i = 2, \ldots, \tilde{n} + 1, \tag{6}$$

where

$$M = \begin{pmatrix} 0 & M_{12} & -M_{12} & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -(M_{22} + M_{23} + M_{33}) & M_{33} & -M_{24} \\ 0 & M_{42} & -M_{42} & 0 \end{pmatrix} \tag{7}$$

and N, L *have the same structure as the matrix* M.

Proof. The proof needs some previous propositions about the structure of the complementary matrices in the expanded system in the new basis. For more details see [2,3]. □

Corollary 5. *Suppose that Theorem* 4 *holds. Then*, *the matrices* $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ *have the form*:

$$\tilde{A} = \begin{pmatrix} A_{11} & \frac{1}{2}A_{12} + M_{12} & \frac{1}{2}A_{12} - M_{12} & A_{13} \\ A_{21} + M_{21} & \frac{1}{2}A_{22} + M_{22} & \frac{1}{2}A_{22} + M_{23} & A_{23} + M_{24} \\ A_{21} - M_{21} & \frac{1}{2}A_{22} - (M_{22} + M_{23} + M_{33}) & \frac{1}{2}A_{22} + M_{33} & A_{23} - M_{24} \\ A_{31} & \frac{1}{2}A_{32} + M_{42} & \frac{1}{2}A_{32} - M_{42} & A_{33} \end{pmatrix}. \tag{8}$$

The matrices $\tilde{B}$ *and* $\tilde{C}$ *have the same structure as* $\tilde{A}$ *when substituting* A_{ij} *by* B_{ij}, M_{ij} *by* N_{ij} *and* A_{ij} *by* C_{ij}, M_{ij} *by* L_{ij}, $i, j = 1, \ldots, 4$, *respectively.*

Proof. By (2), $\tilde{A} = VAU + M$ and by using the matrix A given in (4) and the matrix M of (7), we prove the corollary for the matrix $\tilde{A}$. The same for $\tilde{B}$ and $\tilde{C}$. □

2.3. Computational aspects

Theorem 4 gives us a block structure for the complementary matrices as well as conditions (6) to be satisfied by the blocks to guarantee the Inclusion Principle. In this section, we use these conditions to identify possible choices of the complementary matrices. The choice of these matrices is very important from the computational and control design viewpoint. We may identify two particular classes from (6): (a) when its column submatrices are zero matrices; (b) when its row submatrices are zero matrices. The case (a) results in the following complementary submatrices:

$$M_{12} = 0, \quad M_{23} + M_{33} = 0, \quad M_{42} = 0,$$
$$L_{12} = 0, \quad L_{23} + L_{33} = 0, \quad L_{42} = 0. \tag{9}$$

The case (b) results in the following complementary submatrices:

$$M_{21} = 0, \quad M_{22} + M_{23} = 0, \quad M_{24} = 0,$$
$$N_{21} = 0, \quad N_{22} + N_{23} = 0, \quad N_{24} = 0. \tag{10}$$

The case (a) includes the expansions corresponding to aggregations and the case (b) includes the restrictions [9].

2.4. *The problem*

The motivation of this study is in the missing knowledge on the controllability–observability of subsystems in the expanded space when considering overlapping subsystems. This knowledge is an essential requirement mainly when designing decentralized controllers for the expanded systems. Simultaneously, it is important to know if such an expanded composite system is controllable–observable provided that the initial system holds these properties both on the level of its disjoint subsystems and the composite system level. Particularly, the practical importance of a complete transmission of these qualitative properties from the original system to its expanded form, for both the subsystems and the overall system, follows from the following well known use of the expansion–contraction process: an initial system composed by strongly coupled subsystems is expanded to a system with weakly coupled subsystems and then well developed weak coupling control design methods can be applied. Therefore, one of the basic natural requirements to construct a fully valued expansion–contraction process is the complete transmission of all qualitative properties from the original system to its expansion.

The lack of such study could be due to the usage of forms of complementary matrices corresponding only with particular cases of aggregations and restrictions. Recently, new forms of complementary matrices have been proposed by Bakule et al. [2,3] and Rossell [9]. They offer sufficient flexibility in the selection of complementary matrices to enable this study.

Therefore, the problem can be formulated as follows: suppose that both the original composite system and its disjoint subsystems given in (1) and (4) are both controllable–observable. Suppose that these systems satisfy Corollary 5 when selecting the complementary matrices given in (9) or (10). The goal is to prove that complementary matrices always exist ensuring that both the expanded composite system and its subsystems are controllable–observable.

3. Main results

First, the existence of complementary matrices is proved ensuring controllability–observability of decoupled subsystems in the expanded space. Then, these results are used to prove the same properties for the composite expanded system. The solution is explicitly derived only for the case (a) given by (9). The case (b) is omitted here, because its solution follows a completely analogous way of derivation.

3.1. Subsystems

Let us consider the system $\mathbf{S}$ given by (4) and its expansion $\tilde{\mathbf{S}}$ denoted by

$$\tilde{\mathbf{S}}: \quad \begin{aligned} \begin{pmatrix}\dot{\tilde{x}}_1\\ \dot{\tilde{x}}_2\end{pmatrix} &= \begin{pmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{pmatrix}\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix} + \begin{pmatrix}\tilde{B}_{11} & \tilde{B}_{12}\\ \tilde{B}_{21} & \tilde{B}_{22}\end{pmatrix}\begin{pmatrix}\tilde{u}_1\\ \tilde{u}_2\end{pmatrix},\\ \begin{pmatrix}\tilde{y}_1\\ \tilde{y}_2\end{pmatrix} &= \begin{pmatrix}\tilde{C}_{11} & \tilde{C}_{12}\\ \tilde{C}_{21} & \tilde{C}_{22}\end{pmatrix}\begin{pmatrix}\tilde{x}_1\\ \tilde{x}_2\end{pmatrix}, \end{aligned} \tag{11}$$

where the matrices

$$\tilde{A} = \begin{pmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ \tilde{A}_{21} & \tilde{A}_{22}\end{pmatrix}, \quad \tilde{B} = \begin{pmatrix}\tilde{B}_{11} & \tilde{B}_{12}\\ \tilde{B}_{21} & \tilde{B}_{22}\end{pmatrix} \quad \text{and} \quad \tilde{C} = \begin{pmatrix}\tilde{C}_{11} & \tilde{C}_{12}\\ \tilde{C}_{21} & \tilde{C}_{22}\end{pmatrix}$$

have appropriate dimensions. Now, $\tilde{\mathbf{S}}$ can be represented as two interconnected subsystems:

$$\begin{aligned} \tilde{\mathbf{S}}_1: \quad & \dot{\tilde{x}}_1 = \tilde{A}_{11}\tilde{x}_1 + \tilde{B}_{11}\tilde{u}_1 + \tilde{A}_{12}\tilde{x}_2 + \tilde{B}_{12}\tilde{u}_2,\\ & \tilde{y}_1 = \tilde{C}_{11}\tilde{x}_1 + \tilde{C}_{12}\tilde{x}_2,\\ \tilde{\mathbf{S}}_2: \quad & \dot{\tilde{x}}_2 = \tilde{A}_{22}\tilde{x}_2 + \tilde{B}_{22}\tilde{u}_2 + \tilde{A}_{21}\tilde{x}_1 + \tilde{B}_{21}\tilde{u}_1,\\ & \tilde{y}_2 = \tilde{C}_{22}\tilde{x}_2 + \tilde{C}_{21}\tilde{x}_1, \end{aligned} \tag{12}$$

where $\tilde{A}_{ii}$, $\tilde{B}_{ii}$ and $\tilde{C}_{ii}$ for $i = 1, 2$ are the matrices corresponding to the two decoupled subsystems:

$$\begin{aligned} \tilde{\mathbf{S}}_{\mathbf{D}}^1: \quad & \dot{\tilde{x}}_1 = \tilde{A}_{11}\tilde{x}_1 + \tilde{B}_{11}\tilde{u}_1,\\ & \tilde{y}_1 = \tilde{C}_{11}\tilde{x}_1,\\ \tilde{\mathbf{S}}_{\mathbf{D}}^2: \quad & \dot{\tilde{x}}_2 = \tilde{A}_{22}\tilde{x}_2 + \tilde{B}_{22}\tilde{u}_2,\\ & \tilde{y}_2 = \tilde{C}_{22}\tilde{x}_2, \end{aligned} \tag{13}$$

that is, when we suppose that the matrices $\tilde{A}_{ij}, \tilde{B}_{ij}, \tilde{C}_{ij}, i, j = 1, 2, i \neq j$, are 0. We use constructively the well-known Hautus lemma [13, Lemma 3.3.7]. It asserts that a subsystem $\tilde{\mathbf{S}}_{\mathbf{D}}^i$ in (13) is controllable and observable if and only if

$$\operatorname{rank}(\tilde{A}_{ii} - \lambda I \mid \tilde{B}_{ii}) \quad \text{and} \quad \operatorname{rank}\begin{pmatrix} \tilde{A}_{ii} - \lambda I \\ --- \\ \tilde{C}_{ii} \end{pmatrix}$$

have the same dimension as $\tilde{A}_{ii}$, respectively, for $i = 1, 2$ and for all $\lambda \in \mathbb{C}$. Now, we derive the results on controllability and observability for the decoupled subsystems given in (13).

Theorem 6. *Consider the systems* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *with forms* (4) *and* (11), *respectively. Suppose that the expanded system* $\tilde{\mathbf{S}}$ *satisfies Corollary* 5 *and* (9). *Suppose that the subsystems* (A_{11}, B_{11}) *and* (C_{11}, A_{11}) *in* (4) *are controllable–observable. Then, there always exist submatrices* M_{ij}, N_{ij}, L_{ij} *ensuring that the subsystem* $\tilde{\mathbf{S}}^1_{\mathbf{D}}$ *in* (13) *is controllable–observable.*

Proof. *Controllability*. The matrix of controllability has the following form:

$$\tilde{H}^{\mathrm{c}}_{(\tilde{A}_{11}, \tilde{B}_{11})} = \begin{pmatrix} A_{11} - \lambda I_{n_1} & \frac{1}{2}A_{12} & | & B_{11} & \frac{1}{2}B_{12} + N_{12} \\ & & | & & \\ A_{21} + M_{21} & \frac{1}{2}A_{22} + M_{22} - \lambda I_{n_2} & | & B_{21} + N_{21} & \frac{1}{2}B_{22} + N_{22} \end{pmatrix}. \quad (14)$$

We shall consider two cases: (i) when λ is an eigenvalue of A_{11} and (ii) otherwise.

Case (i). Suppose λ is an eigenvalue of A_{11}. Select the columns of the matrix $(A_{11} - \lambda I_{n_1} \mid B_{11})$ that give rank n_1 for a given eigenvalue λ of A_{11}. This is possible because we assume that the initial subsystem (A_{11}, B_{11}) is controllable. Denote this new matrix as Q_1. The corresponding columns of the matrices $(A_{21} + M_{21})$ and $(B_{21} + N_{21})$ form a new block matrix denoted by Q_3. The remaining nonselected columns of $(A_{11} - \lambda I_{n_1} \mid B_{11})$ with the corresponding columns of $(A_{21} + M_{21})$ and $(B_{21} + N_{21})$ are joined to the matrices $(\frac{1}{2}B_{12} + N_{12})$ and $(\frac{1}{2}B_{22} + N_{22})$, respectively, in order to form two block matrices denoted by Q_2 and Q_4, respectively. Thus, we obtain

$$\operatorname{rank} \tilde{H}^{\mathrm{c}}_{(\tilde{A}_{11}, \tilde{B}_{11})} = \operatorname{rank}\begin{pmatrix} Q_1 & | & Q_2 & | & \frac{1}{2}A_{12} \\ - & - & - & - & - \\ Q_3 & | & Q_4 & | & \frac{1}{2}A_{22} + M_{22} - \lambda I_{n_2} \end{pmatrix}.$$

Consider $F_1 = Q_1^{-1} Q_2$ and $F_2 = \frac{1}{2} Q_1^{-1} A_{12}$. Consider the matrix

$$\begin{pmatrix} -F_1 & -F_2 & I_{n_1} \\ I_{m_1+m_2} & 0 & 0 \\ 0 & I_{n_2} & 0 \end{pmatrix}.$$

Then

$$\left(\begin{array}{c|c|c} Q_1 & Q_2 & \frac{1}{2}A_{12} \\ \hline Q_3 & Q_4 & \frac{1}{2}A_{22}+M_{22}-\lambda I_{n_2} \end{array}\right)\begin{pmatrix} -F_1 & -F_2 & I_{n_1} \\ I_{m_1+m_2} & 0 & 0 \\ 0 & I_{n_2} & 0 \end{pmatrix}$$

$$= \left(\begin{array}{c|c|c} 0 & 0 & Q_1 \\ \hline -Q_3F_1+Q_4 & -Q_3F_2+\frac{1}{2}A_{22}+M_{22}-\lambda I_{n_2} & Q_3 \end{array}\right).$$

This matrix has the same rank as $\tilde{H}^{\mathrm{c}}_{(\tilde{A}_{11},\tilde{B}_{11})}$. Consider $M_{22} = Q_3F_2 - \frac{1}{2}A_{22} + X$ with a free matrix X. The columns of the matrix $(A_{11} - \lambda I_{n_1} \mid B_{11})$ that give rank n_1 are not the same for all eigenvalues λ of A_{11}. This makes that the matrix Q_3 depends on λ and thereby also M_{22} will be dependent on λ. To eliminate this nondesirable dependence, we simply select $M_{21} = -A_{21}$ and $N_{21} = -B_{21}$ so that $Q_3 = 0$. Now we select the matrix X as a diagonal matrix with distinct eigenvalues, which are simultaneously different from those ones of A_{11}, A_{33} and also satisfying that the eigenvalues of $(2X - A_{22})$ are distinct and different from those ones of the matrix A of the initial overall system. This is not restrictive, such a selection always exists. Then, $M_{22} = -\frac{1}{2}A_{22} + X$ and $\operatorname{rank}(-Q_3F_2 + \frac{1}{2}A_{22} + M_{22} - \lambda I_{n_2}) = \operatorname{rank}(X - \lambda I_{n_2}) = n_2$ for all eigenvalues λ of A_{11}. Thus, $\operatorname{rank} \tilde{H}^{\mathrm{c}}_{(\tilde{A}_{11},\tilde{B}_{11})} = n_1 + n_2$ for all eigenvalues λ of A_{11}.

Case (ii): Suppose that λ is not an eigenvalue of A_{11}, but it is an eigenvalue of X. We proceed as in case (i), but now we can select $Q_1 = A_{11} - \lambda I_{n_1}$ with rank n_1. We obtain $Q_2 = (B_{11} \mid \frac{1}{2}B_{12} + N_{12})$ and $Q_4 = (0 \mid \frac{1}{2}B_{22} + N_{22})$ independent of λ. In this case $\operatorname{rank}(X - \lambda I_{n_2}) = n_2 - 1$ for all eigenvalues λ of X. We need to select only one nonzero column of the matrix Q_4 and substitute this column into the matrix $(X - \lambda I_{n_2})$ in order to obtain $\operatorname{rank}(X - \lambda I_{n_2}) = n_2$. The $n_2 - 1$ independent columns of $(X - \lambda I_{n_2})$ are not the same for all λ. Since N_{22} is a completely free matrix, it is possible to obtain one nonzero column of Q_4 with any nonzero values such that its substitution into $(X - \lambda I_{n_2})$ gives rank n_2 for all eigenvalues λ of X. Then, $\operatorname{rank} \tilde{H}^{\mathrm{c}}_{(\tilde{A}_{11},\tilde{B}_{11})} = n_1 + n_2$.

If λ is not an eigenvalue of X, then $\operatorname{rank}(X - \lambda I_{n_2}) = n_2$ for all λ, and consequently $\operatorname{rank} \tilde{H}^{\mathrm{c}}_{(\tilde{A}_{11},\tilde{B}_{11})} = n_1 + n_2$.

Thus, the conditions $M_{21} = -A_{21}$, $N_{21} = -B_{21}$ and an appropriate selection of M_{22} and N_{22} guarantee the controllability.

Observability. The matrix of observability has the following form:

$$\tilde{H}^{\mathrm{o}}_{(\tilde{C}_{11},\tilde{A}_{11})} = \begin{pmatrix} A_{11} - \lambda I_{n_1} & \frac{1}{2}A_{12} \\ A_{21} + M_{21} & \frac{1}{2}A_{22} + M_{22} - \lambda I_{n_2} \\ \hline C_{11} & \frac{1}{2}C_{12} \\ C_{21} + L_{21} & \frac{1}{2}C_{22} + L_{22} \end{pmatrix}. \tag{15}$$

Suppose that the matrices $M_{21} = -A_{21}$, $M_{22} = -\frac{1}{2}A_{22} + X$ have been selected as in the case of controllability. We shall consider the two cases as above.

Case (i). Suppose λ is an eigenvalue of A_{11}. Select the rows of the matrix

$$\begin{pmatrix} A_{11} - \lambda I_{n_1} \\ \hline C_{11} \end{pmatrix}$$

resulting in the rank n_1 for a given eigenvalue λ of A_{11}. It is possible because we assume that the initial subsystem (C_{11}, A_{11}) is observable. Denote this new matrix as Q_5. The corresponding rows of the matrices $\frac{1}{2}A_{12}$ and $\frac{1}{2}C_{12}$ form a new matrix denoted by Q_6. The remaining nonselected rows of the matrix

$$\begin{pmatrix} A_{11} - \lambda I_{n_1} \\ \hline C_{11} \end{pmatrix}$$

with the corresponding rows of $\frac{1}{2}A_{12}$ and $\frac{1}{2}C_{12}$ are joined to $(C_{21} + L_{21})$ and $(\frac{1}{2}C_{22} + L_{22})$, respectively, in order to form two block matrices denoted as Q_7 and Q_8, respectively. Thus,

$$\operatorname{rank} \tilde{H}^{\mathrm{o}}_{(\tilde{C}_{11},\tilde{A}_{11})} = \operatorname{rank} \left(\begin{array}{c|c} Q_5 & Q_6 \\ \hline 0 & X - \lambda I_{n_2} \\ \hline Q_7 & Q_8 \end{array}\right)$$

with $\operatorname{rank} Q_5 = n_1$ and $\operatorname{rank}(X - \lambda I_{n_2}) = n_2$. Therefore, $\operatorname{rank} \tilde{H}^{\mathrm{o}}_{(\tilde{C}_{11},\tilde{A}_{11})} = n_1 + n_2$ for all eigenvalues λ of A_{11}.

Case (ii). Suppose that λ is not an eigenvalue of A_{11}, but it is an eigenvalue of X. We proceed as in the case (i), but with $Q_5 = A_{11} - \lambda I_{n_1}$ with rank n_1. Select $L_{21} = -C_{21}$. We obtain

$$Q_7 = \begin{pmatrix} C_{11} \\ 0 \end{pmatrix} \quad \text{and} \quad Q_8 = \begin{pmatrix} \frac{1}{2}C_{12} \\ \frac{1}{2}C_{22} + L_{22} \end{pmatrix}$$

independent of λ. Then

$$\operatorname{rank} \tilde{H}^{\mathrm{o}}_{(\tilde{C}_{11},\tilde{A}_{11})} = \operatorname{rank} \begin{pmatrix} A_{11} - \lambda I_{n_1} & | & \frac{1}{2}A_{12} \\ \hline 0 & | & X - \lambda I_{n_2} \\ \hline \begin{pmatrix} C_{11} \\ 0 \end{pmatrix} & | & \begin{pmatrix} \frac{1}{2}C_{12} \\ \frac{1}{2}C_{22} + L_{22} \end{pmatrix} \end{pmatrix}.$$

In this case rank $(X - \lambda I_{n_2}) = n_2 - 1$ for all eigenvalues λ of X. We need to select only one nonzero row of the matrix $(\frac{1}{2}C_{22} + L_{22})$ and substitute it into $(X - \lambda I_{n_2})$ such that rank $(X - \lambda I_{n_2}) = n_2$. The $n_2 - 1$ independent rows of $(X - \lambda I_{n_2})$ are not the same for all λ. Since L_{22} is a completely free matrix, it is possible to obtain one row of $(\frac{1}{2}C_{22} + L_{22})$ with any nonzero values such that its substitution into $(X - \lambda I_{n_2})$ gives rank n_2 for all eigenvalues λ of X. Thus, rank $\tilde{H}^{\mathrm{o}}_{(\tilde{C}_{11},\tilde{A}_{11})} = n_1 + n_2$.

If λ is not an eigenvalue of X, then rank $(X - \lambda I_{n_2}) = n_2$ and we obtain rank $\tilde{H}^{\mathrm{o}}_{(\tilde{C}_{11},\tilde{A}_{11})} = n_1 + n_2$.

Therefore, the selection of $L_{21} = -C_{21}$ and an appropriate choice of L_{22} guarantee the observability.

Consequently, there exist complementary submatrices satisfying controllability–observability simultaneously. □

Theorem 7. *Consider the systems* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *with forms* (4) *and* (11), *respectively. Suppose that the expanded system* $\tilde{\mathbf{S}}$ *satisfies Corollary* 5 *and* (9). *Suppose that the subsystems* (A_{33}, B_{33}) *and* (C_{33}, A_{33}) *in* (4) *are controllable–observable. Then, there always exist submatrices* M_{ij}, N_{ij}, L_{ij} *ensuring that the subsystem* $\tilde{\mathbf{S}}^2_{\mathbf{D}}$ *in* (13) *is controllable–observable.*

Proof. *Controllability*. The proof follows similar steps as those ones in the proof of Theorem 6. The matrix of controllability has the form

$$\tilde{H}^{\mathrm{c}}_{(\tilde{A}_{22},\tilde{B}_{22})} = \begin{pmatrix} \frac{1}{2}A_{22} + M_{33} - \lambda I_{n_2} & A_{23} - M_{24} & | & \frac{1}{2}B_{22} + N_{33} & B_{23} - N_{24} \\ & & | & & \\ \frac{1}{2}A_{32} & A_{33} - \lambda I_{n_3} & | & \frac{1}{2}B_{32} - N_{42} & B_{33} \end{pmatrix}. \quad (16)$$

We shall consider two cases: (i) when λ is an eigenvalue of A_{33} and (ii) otherwise.

Case (i). Suppose λ is an eigenvalue of A_{33}. Select the columns of the matrix $(A_{33} - \lambda I_{n_3} \mid B_{33})$ that give rank n_3 for a given λ of A_{33}. This is possible because we assume that the initial subsystem (A_{33}, B_{33}) is controllable. We denote this new matrix as Q_{11}. The corresponding columns of the matrices $(A_{23} - M_{24})$ and $(B_{23} - N_{24})$ form a new block matrix denoted by Q_9. The remaining nonselected columns of the matrices $(A_{33} - \lambda I_{n_3} \mid B_{33})$ together with the corresponding columns of $(A_{23} - M_{24})$ and $(B_{23} - N_{24})$ are joined to the matrices $(\frac{1}{2}B_{32} - N_{42})$ and $(\frac{1}{2}B_{22} + N_{33})$, respectively, in order to form two block matrices denoted by Q_{12} and Q_{10}, respectively. Thus,

$$\operatorname{rank} \tilde{H}^{\mathrm{c}}_{(\tilde{A}_{22},\tilde{B}_{22})} = \operatorname{rank} \left(\begin{array}{c|c|c} \frac{1}{2}A_{22} + M_{33} - \lambda I_{n_2} & Q_9 & Q_{10} \\ \hline \frac{1}{2}A_{32} & Q_{11} & Q_{12} \end{array}\right).$$

Since Q_{11} is a nonsingular matrix, there exist $F_3 = Q_{11}^{-1}Q_{12}$ and $F_4 = \frac{1}{2}Q_{11}^{-1}A_{32}$. Consider the matrix

$$\begin{pmatrix} 0 & I_{n_2} & 0 \\ -F_3 & -F_4 & I_{n_3} \\ I_{m_2+m_3} & 0 & 0 \end{pmatrix}.$$

Therefore, we obtain

$$\begin{aligned} &\left(\begin{array}{c|c|c} \frac{1}{2}A_{22} + M_{33} - \lambda I_{n_2} & Q_9 & Q_{10} \\ \hline \frac{1}{2}A_{32} & Q_{11} & Q_{12} \end{array}\right) \begin{pmatrix} 0 & I_{n_2} & 0 \\ -F_3 & -F_4 & I_{n_3} \\ I_{m_2+m_3} & 0 & 0 \end{pmatrix} \\ &\quad = \left(\begin{array}{c|c|c} -Q_9F_3 + Q_{10} & -Q_9F_4 + \frac{1}{2}A_{22} + M_{33} - \lambda I_{n_2} & Q_9 \\ \hline 0 & 0 & Q_{11} \end{array}\right). \end{aligned}$$

This matrix has the same rank as $\tilde{H}^{\mathrm{c}}_{(\tilde{A}_{22},\tilde{B}_{22})}$. Consider that $M_{33} = Q_9F_4 - \frac{1}{2}A_{22} + X$, where X is the same matrix as selected in the proof of controllability of Theorem 6. The columns of the matrix $(A_{33} - \lambda I_{n_3} \mid B_{33})$ that give rank n_3 are not the same for all eigenvalues λ of A_{33}. This makes that the matrix Q_9 depends on λ and thereby also M_{33} will be dependent on λ. To eliminate this dependence, we simply select $M_{24} = A_{23}$ and $N_{24} = B_{23}$, so that $Q_9 = 0$. Then, $M_{33} = -\frac{1}{2}A_{22} + X$ and $\operatorname{rank}(-Q_9F_4 + \frac{1}{2}A_{22} + M_{33} - \lambda I_{n_2}) = \operatorname{rank}(X - \lambda I_{n_2}) = n_2$ for all eigenvalues λ of A_{33}. Thus, $\operatorname{rank} \tilde{H}^{\mathrm{c}}_{(\tilde{A}_{22},\tilde{B}_{22})} = n_2 + n_3$. This procedure holds for all eigenvalues λ of A_{33}.

Case (ii). Suppose that λ is not an eigenvalue of A_{33}, but it is an eigenvalue of X. We proceed as in the above case (i), but now we can use $Q_{11} = A_{33} - \lambda I_{n_3}$ with rank n_3. We obtain $Q_{10} = (\frac{1}{2}B_{22} + N_{33} \mid 0)$ and $Q_{12} = (\frac{1}{2}B_{32} - N_{42} \mid B_{33})$ independent of λ. In this case $\operatorname{rank}(X - \lambda I_{n_2}) = n_2 - 1$ for all eigenvalues λ of X. We need to select only one nonzero column of the matrix $Q_{10} = (\frac{1}{2}B_{22} + N_{33} \mid 0)$ and substitute this column into the matrix $(X - \lambda I_{n_2})$ in order to obtain $\operatorname{rank}(X - \lambda I_{n_2}) = n_2$. The $n_2 - 1$ independent columns of $(X - \lambda I_{n_2})$ are not the same for all λ. Since N_{33} is a completely free matrix, it is possible to obtain one column of $(\frac{1}{2}B_{22} + N_{33})$ with any nonzero values such that its substitution into $(X - \lambda I_{n_2})$ gives $\operatorname{rank} n_2$ for all eigenvalues λ of X. Then, $\operatorname{rank} \tilde{H}^{\mathrm{c}}_{(\tilde{A}_{22},\tilde{B}_{22})} = n_2 + n_3$.

If λ is not an eigenvalue of X, then $\operatorname{rank}(X - \lambda I_{n_2}) = n_2$ for all λ, and therefore $\operatorname{rank} \tilde{H}^{\mathrm{c}}_{(\tilde{A}_{22},\tilde{B}_{22})} = n_2 + n_3$.

Thus, the conditions $M_{24} = A_{23}$, $N_{24} = B_{23}$ and an appropriate selection of M_{33} and N_{33} guarantees the controllability.

Observability. The matrix of observability has the following form:

$$\tilde{H}^{o}_{(\tilde{C}_{22},\tilde{A}_{22})} = \begin{pmatrix} \frac{1}{2}A_{22} + M_{33} - \lambda I_{n_2} & A_{23} - M_{24} \\ \frac{1}{2}A_{32} & A_{33} - \lambda I_{n_3} \\ \hline \frac{1}{2}C_{22} + L_{33} & C_{23} - L_{24} \\ \frac{1}{2}C_{32} & C_{33} \end{pmatrix}. \tag{17}$$

Suppose that $M_{24} = A_{23}$ and $M_{33} = -\frac{1}{2}A_{22} + X$. We shall consider the two cases as above.

Case (i). Suppose λ is an eigenvalue of A_{33}. We select the rows of the matrix

$$\begin{pmatrix} A_{33} - \lambda I_{n_3} \\ \hline C_{33} \end{pmatrix}$$

resulting in the rank n_3 for a given eigenvalue λ of A_{33}. It is possible because we assume that the initial subsystem (C_{33}, A_{33}) is observable. We denote this new matrix as Q_{14}. The corresponding rows of the matrices $\frac{1}{2}A_{32}$ and $\frac{1}{2}C_{32}$ form a new block matrix denoted by Q_{13}. The remaining nonselected rows of

$$\begin{pmatrix} A_{33} - \lambda I_{n_3} \\ \hline C_{33} \end{pmatrix}$$

together with the corresponding rows of $\frac{1}{2}A_{32}$ and $\frac{1}{2}C_{32}$ are joined to the matrices $(C_{23} - L_{24})$ and $(\frac{1}{2}C_{22} + L_{33})$, respectively, in order to form two block matrices denoted as Q_{16} and Q_{15}, respectively. Thus,

$$\operatorname{rank} \tilde{H}^{o}_{(\tilde{C}_{22},\tilde{A}_{22})} = \operatorname{rank} \left(\begin{array}{c|c} X - \lambda I_{n_2} & 0 \\ \hline Q_{13} & Q_{14} \\ \hline Q_{15} & Q_{16} \end{array}\right)$$

with $\operatorname{rank}(X - \lambda I_{n_2}) = n_2$, $\operatorname{rank} Q_{14} = n_3$ and so that $\operatorname{rank} \tilde{H}^{o}_{(\tilde{C}_{22},\tilde{A}_{22})} = n_2 + n_3$. This procedure holds for all eigenvalues λ of A_{33}.

Case (ii). Consider that λ is not an eigenvalue of A_{33}, but it is an eigenvalue of X. We proceed as in case (i), but now we can use $Q_{14} = A_{33} - \lambda I_{n_3}$ with rank n_3 for all λ. We also have

$$Q_{15} = \begin{pmatrix} \frac{1}{2}C_{22} + L_{33} \\ \frac{1}{2}C_{32} \end{pmatrix} \quad \text{and} \quad Q_{16} = \begin{pmatrix} C_{23} - L_{24} \\ C_{33} \end{pmatrix}$$

independent of λ. In this case $\operatorname{rank}(X - \lambda I_{n_2}) = n_2 - 1$ for all eigenvalues λ of X. We select $L_{24} = C_{23}$. We need to select one row of the matrix $(\frac{1}{2}C_{22} + L_{33})$ and substitute it into $(X - \lambda I_{n_2})$ such that rank $(X - \lambda I_{n_2}) = n_2$. The $n_2 - 1$ independent rows of $(X - \lambda I_{n_2})$ are not the same for all λ. Since L_{33} is a completely free matrix, it is possible to obtain one row of $(\frac{1}{2}C_{22} + L_{33})$ with any nonzero values such that its substitution into $(X - \lambda I_{n_2})$ gives rank n_2 for all eigenvalues λ of X. Thus, rank $\tilde{H}^{\mathrm{o}}_{(\tilde{C}_{22},\tilde{A}_{22})} = n_2 + n_3$.

If λ is not eigenvalue of X, then rank $(X - \lambda I_{n_2}) = n_2$ for all λ, and then rank $\tilde{H}^{\mathrm{o}}_{(\tilde{C}_{22},\tilde{A}_{22})} = n_2 + n_3$.

Therefore, we need only to select $L_{24} = C_{23}$ and an appropriate matrix L_{33} to ensure the observability.

Consequently, the simultaneous choice of the complementary submatrices satisfying controllability–observability is guaranteed. □

Remark. We must select the complementary submatrices $M_{21} = -A_{21}$, $N_{21} = -B_{21}$, $L_{21} = -C_{21}$ and M_{22} as follows from the proof of Theorem 6. We also have to select one column of N_{22} and one row of L_{22}. Further, the complementary submatrices to be selected are $M_{24} = A_{23}$, $N_{24} = B_{23}$, $L_{24} = C_{23}$ and M_{33} as follows from the proof of Theorem 7. We also have to select one column of N_{33} and one row of L_{33}. Thus, there do not exist repeated complementary submatrices in the subsystems $\tilde{\mathbf{S}}^1_{\mathbf{D}}$ and $\tilde{\mathbf{S}}^2_{\mathbf{D}}$. Therefore, we have a sufficient freedom in selection of independent complementary submatrices in order to guarantee simultaneous controllability–observability for both subsystems $\tilde{\mathbf{S}}^1_{\mathbf{D}}$ and $\tilde{\mathbf{S}}^2_{\mathbf{D}}$.

3.2. Composite system

Suppose that the controllability–observability of subsystems is proved. It means that we start to prove these properties for the overall expanded system $\tilde{\mathbf{S}}$ with a particular selection of some complementary matrices as given above.

Theorem 8. *Consider the systems* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *with forms* (4) *and* (11), *respectively. Suppose that the initial system* $\mathbf{S}$ *is controllable–observable and the expanded system* $\tilde{\mathbf{S}}$ *satisfies Corollary* 5 *and* (9). *Suppose that the complementary submatrices have been selected to satisfy Theorems* 6 *and* 7 *such that the subsystems* $\tilde{\mathbf{S}}^1_{\mathbf{D}}$ *and* $\tilde{\mathbf{S}}^2_{\mathbf{D}}$ *in* (13) *are controllable–observable. Then, the expanded composite system* $\tilde{\mathbf{S}}$ *is controllable–observable.*

Proof. *Controllability*. By doing manipulations with the initial rows and columns of the expanded matrix $(\tilde{A} - \lambda I_{\tilde{n}} \mid \tilde{B})$ given by Corollary 5 and satisfying (9) with the particular selection of the complementary matrices used in Theorems 6 and 7 the matrix of controllability $\tilde{H}^{\mathrm{c}}_{(\tilde{A},\tilde{B})}$ can be transformed into the form

$$\tilde{H}^{\mathrm{c}}_{(\tilde{A},\tilde{B})}=\left(\begin{array}{ccc|ccc|c|c} A_{11}-\lambda I_{n_1} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} & N_{12} & 0 \\ A_{21} & A_{22}-\lambda I_{n_2} & A_{23} & B_{21} & B_{22} & B_{23} & -\frac{1}{2}N_{23}-\frac{1}{2}N_{33} & 0 \\ A_{31} & A_{32} & A_{33}-\lambda I_{n_3} & B_{31} & B_{32} & B_{33} & N_{42} & 0 \\ \hline 2A_{21} & 2A_{22}-2X & 0 & 2B_{21} & B_{22}-N_{22}-N_{23} & 0 & -\frac{1}{2}N_{22}-\frac{1}{2}N_{23}-N_{33} & 2X-A_{22}-\lambda I_{n_2} \end{array}\right). \tag{18}$$

We denote this matrix as

$$\tilde{H}^{\mathrm{c}}_{(\tilde{A},\tilde{B})}=\left(\begin{array}{c|c|c|c} A-\lambda I_n & B & P_1 & 0 \\ \hline P_2 & P_3 & P_4 & 2X-A_{22}-\lambda I_{n_2} \end{array}\right).$$

We shall consider two cases: (i) when λ is an eigenvalue of A and (ii) otherwise.

Case (i). Suppose λ is an eigenvalue of A. Select the columns of the matrix $(A-\lambda I_n|B)$ that give rank n for a given eigenvalue λ of A. This is possible because we assume that the initial system (A, B) is controllable. We denote this new matrix as P_5. The corresponding columns of the matrices P_2 and P_3 form a matrix that is denoted by P_7. The remaining nonselected columns of the matrices $(A-\lambda I_n|B)$ together with the corresponding columns of the matrices P_2 and P_3 are joined to the matrices P_1 and P_4, respectively, in order to form two block matrices denoted by P_6 and P_8, respectively. Thus,

$$\operatorname{rank}\tilde{H}^{\mathrm{c}}_{(\tilde{A},\tilde{B})}=\operatorname{rank}\left(\begin{array}{c|c|c} P_5 & P_6 & 0 \\ \hline P_7 & P_8 & 2X-A_{22}-\lambda I_{n_2} \end{array}\right).$$

Then, P_5 has rank n and rank $(2X-A_{22}-\lambda I_{n_2})=n_2$. Consequently, rank $\tilde{H}^{\mathrm{c}}_{(\tilde{A},\tilde{B})}=n+n_2=\tilde{n}$ for all eigenvalues λ of A.

Case (ii). Now suppose that λ is not an eigenvalue of A, but it is an eigenvalue of $(2X-A_{22})$. Consider the above matrix of controllability

$$\tilde{H}^{\mathrm{c}}_{(\tilde{A},\tilde{B})}=\left(\begin{array}{c|c|c|c} A-\lambda I_n & B & P_1 & 0 \\ \hline P_2 & P_3 & P_4 & 2X-A_{22}-\lambda I_{n_2} \end{array}\right).$$

Since λ is not an eigenvalue of A, rank $(A-\lambda I_n)=n$. In this case, rank $(2X-A_{22}-\lambda I_{n_2})=n_2-1$ for all λ. We select $N_{12}=0$, $N_{23}=-N_{33}$, $N_{42}=0$. Then, $P_1=0$ and $P_4=-\frac{1}{2}N_{22}-\frac{1}{2}N_{33}$. We need to select only one column of P_4 and substitute it into $(2X-A_{22}-\lambda I_{n_2})$ such that rank $(2X-A_{22}-\lambda I_{n_2})=n_2$. The n_2-1 independent columns of $(2X-A_{22}-\lambda I_{n_2})$ are not the same for all λ. One column of N_{22} and one column of N_{33} have been required to be selected for the controllability of the subsystems (Theorems 6 and 7). But enough freedom exists in these matrices to get one column of P_4 such that the above rank n_2 condition is satisfied for all the eigenvalues λ of $(2X - A_{22})$. Thus, rank $\tilde{H}^{\mathrm{c}}_{(\tilde{A},\tilde{B})} = n + n_2 = \tilde{n}$.

If λ is not an eigenvalue of $(2X - A_{22})$, then we obtain rank $(A - \lambda I_n) = n$ and rank $(2X - A_{22} - \lambda I_{n_2}) = n_2$. Consequently, $\tilde{H}^{\mathrm{c}}_{(\tilde{A},\tilde{B})} = n + n_2 = \tilde{n}$.

Observability. By doing manipulations with the initial rows and columns of the expanded matrix

$$\begin{pmatrix} \tilde{A} - \lambda I_{\tilde{n}} \\ --- \\ \tilde{C} \end{pmatrix}$$

given by Corollary 5 and satisfying (9) with the particular selection of the complementary matrices used in Theorems 6, 7 the matrix of observability $\tilde{H}^{\mathrm{o}}_{(\tilde{C},\tilde{A})}$ can be transformed into the form

$$\tilde{H}^{\mathrm{o}}_{(\tilde{C},\tilde{A})} = \left(\begin{array}{ccc|c} A_{11} - \lambda I_{n_1} & A_{12} & A_{13} & 0 \\ A_{21} & A_{22} - \lambda I_{n_2} & A_{23} & 0 \\ A_{31} & A_{32} & A_{33} - \lambda I_{n_3} & 0 \\ \hline C_{11} & C_{12} & C_{13} & 0 \\ C_{21} & C_{22} & C_{23} & 0 \\ C_{31} & C_{32} & C_{33} & 0 \\ \hline 2C_{21} & C_{22} + 2L_{33} & 0 & -L_{22} - L_{33} \\ \hline 0 & 2A_{22} - 2X & 2A_{23} & 2X - A_{22} - \lambda I_{n_2} \end{array}\right). \tag{19}$$

We denote this matrix as

$$\tilde{H}^{\mathrm{o}}_{(\tilde{C},\tilde{A})} = \left(\begin{array}{c|c} A - \lambda I_n & 0 \\ \hline C & 0 \\ \hline P_9 & -L_{22} - L_{33} \\ \hline P_{10} & 2X - A_{22} - \lambda I_{n_2} \end{array}\right). \tag{20}$$

We shall consider the two cases as above.

Case (i). Suppose λ is an eigenvalue of A. Select the rows of

$$\begin{pmatrix} A - \lambda I_n \\ --- \\ C \end{pmatrix}$$

resulting in the rank n for a given λ. This is possible because we assume that the initial system (C, A) is observable. We denote this new matrix as P_{11}. The remaining nonselected rows of the matrix

$$\begin{pmatrix} A-\lambda I_n \\ --- \\ C \end{pmatrix}$$

together with the matrix P_9 and $(-L_{22}-L_{33})$ are joined in two block matrices denoted as P_{12} and P_{13}, respectively. Thus,

$$\operatorname{rank} \tilde{H}^{\mathrm{o}}_{(\tilde{C},\tilde{A})} = \operatorname{rank} \begin{pmatrix} P_{11} \mid & 0 \\ ----- & ----- \\ P_{12} \mid & P_{13} \\ ----- & ----- \\ P_{10} \mid & 2X-A_{22}-\lambda I_{n_2} \end{pmatrix}.$$

Since P_{11} is a nonsingular matrix, rank $P_{11}=n$ and rank $(2X-A_{22}-\lambda I_{n_2})=n_2$. Then, $\tilde{H}^{\mathrm{o}}_{(\tilde{C},\tilde{A})}=n+n_2=\tilde{n}$ for all eigenvalues λ of A.

Case (ii). Now suppose that λ is not an eigenvalue of A, but it is an eigenvalue of $(2X-A_{22})$. We consider the observability matrix in (20), where now rank $(A-\lambda I_n)=n$ and rank $(2X-A_{22}-\lambda I_{n_2})=n_2-1$ for all λ. We need only one row of $(P_9 \mid -L_{22}-L_{33})$ to be substituted into $(P_{10} \mid 2X-A_{22}-\lambda I_{n_2})$ such that rank $(2X-A_{22}-\lambda I_{n_2})=n_2$. The n_2-1 independent rows of $(2X-A_{22}-\lambda I_{n_2})$ are not the same for all λ. One row of L_{22} and one row of L_{33} have been required to be selected to ensure observability of the subsystems (Theorems 6 and 7). But there is enough freedom in these matrices to obtain one row of $(-L_{22}-L_{33})$ such that the above rank n_2 condition is fulfilled for all eigenvalues λ of $(2X-A_{22})$. Thus, rank $\tilde{H}^{\mathrm{c}}_{(\tilde{A},\tilde{B})}=n+n_2=\tilde{n}$.

If λ is not an eigenvalue of $(2X-A_{22})$, then rank $(2X-A_{22}-\lambda I_{n_2})=n_2$. Thus, $\tilde{H}^{\mathrm{o}}_{(\tilde{C},\tilde{A})}=n+n_2=\tilde{n}$.

Consequently, the simultaneous choice of the complementary submatrices satisfying both controllability–observability properties is always possible. □

Remark. Besides the selections required in Theorems 6 and 7, we have selected $N_{12}=0$, $N_{23}=-N_{33}$ and $N_{42}=0$ as follows from the proof of Theorem 8. We have to select also one column of matrix $P_4=-\frac{1}{2}N_{22}-\frac{1}{2}N_{33}$ and one row of matrix $(-L_{22}-L_{33})$. We have also introduced conditions on the matrix X as a diagonal matrix with distinct eigenvalues, which are simultaneously different from those ones of A_{11}, A_{33} and also satisfying that the eigenvalues of $(2X-A_{22})$ are distinct and different from those ones of A. This is not at all restrictive. The selection of X satisfying these conditions is always possible and it is not difficult to impose them into the practical usage.

4. Conclusion

Simultaneous preservation of controllability–observability of both the composite system and its subsystems in expanded space within the Inclusion Principle has been studied under the assumption that both the disjoint subsystems and the overall composite original system are controllable–observable. To solve this problem, two recently identified broad classes of complementary matrices have been used. They offer sufficient flexibility in the selection of complementary matrices to solve this problem. It has been proved that complementary matrices always exist within these classes ensuring that both the expanded composite system and its subsystems are controllable–observable. The selection of these matrices is simple in practice. This result extends in an original way the current knowledge concerning controllability–observability within the Inclusion Principle in that results on simultaneous preservation of controllability–observability of subsystems and the composite system in the expanded space have not been available up to now. It is expected that this result will be useful mainly in the design of overlapping decentralized controllers.

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